14.7.1 Find the local maxima, local minima, and saddle points of the function.

$f(x,y) = x^2 + xy + y^2 + 3x - 3y + 5$

1. Find point of interest (critical point) when $f_x = 0$ and $f_y = 0$

2. Find D (discriminant) $f_{xx}f_{yy} - f_{xy}^2$

THEOREM 11—Second Derivative Test for Local Extreme Values
Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

1. $f_{xx} < 0$, $D > 0$, $f(a,b) \to$ local maximum "
2. $f_{xx} > 0$, $D > 0$, $f(a,b) \to$ local minimum. "
3. $D < 0$, $f(a,b) \to$ saddle point "

1

$f_x = 2x + y + 3$ $\qquad 2x + y + 3 = 0$

$f_y = x + 2y - 3$ $\qquad x + 2y - 3 = 0$

solve $f_y = 0$ for x.

$x + 2y - 3 = 0$

$x = -2y + 3$

substitute $x = -2y + 3$ in the equation f_x and solve for y.

$2x + y + 3 = 0$

$2(-2y + 3) + y + 3 = 0$

$-4y + 6 + y + 3 = 0$

$-3y = -9$

$y = 3$

substitute $y = 3$ in the equation for x.

$x = -2y + 3$

$= -2(3) + 3$

$x = -3$

thus, $(-3,3)$ is a critical point.

2

$f_x = 2x + y + 3$

$f_{xx} = 2$

$f_y = x + 2y - 3$

$f_{yy} = 2$

differenciate f_x with respect to y.

$f_{xy} = 2x + y + 3$

$= 1$

$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 1^2$

$= 3$

since $f_x > 0$ and $D > 0$ f has local minimum.
at $(-3, 3)$

evaluate $f(-3,3)$

$f(-3,3) = x^2 + xy + y^2 + 3x - 3y + 5$

$= (-3)^2 + (-3)(3) + (3)^2 + 3(-3) - 3(3) + 5$

$= -4$

14.7.1 find the local maxima, local minima, and saddle points of the function.

$f(x,y) = x^2 + xy + y^2 + 6x - 6y + 6$

1. Find point of interest (critical point) when $f_x = 0$ and $f_y = 0$

$f_x = 2x + y + 6 \qquad 2x + y + 6 = 0$

$f_y = x + 2y - 6 \qquad x + 2y - 6 = 0$

solve $f_y = 0$ for x.

$x + 2y - 6 = 0$

$x = -2y + 6$

substitute $x = -2y + 6$ in the equation f_x and solve for y.

$2x + y + 6 = 0$

$2(-2y + 6) + y + 6 = 0$

$-4y + 12 + y + 6 = 0$

$-3y = -18$

$y = 6$

substitute $y = 6$ in the equation for x.

$x = -2y + 6$

$= -2(6) + 6$

$x = -6$

thus. $(-6, 6)$ is a critical point.

2. Find D (discriminant)

$f_x = 2x + y + 6$

$f_{xx} = 2$

$f_y = x + 2y - 6$

$f_{yy} = 2$

differenciate f_x with respect to y.

$f_{xy} = 2x + y + 6$

$= 1$

$f_{xx} f_{yy} - f_{xy^2} = (2)(2) - 1^2$

$= 3$

evaluate $f(-6,6)$

$f(-6,6) = x^2 + xy + y^2 + 6x - 6y + 6$

$= (-6)^2 + (-6)(6) + (6)^2 + 6(-6) - 6(6) + 6$

$= -30$

since $f_{xx} > 0$, $D > 0$, $f(-6,6) \rightarrow$ local minimum with value of -30.

14.7.3 Find the local maxima, local minima, and saddle points of the function.

$f(x,y) = x^2 + xy + 4x + 5y - 4$

1. Find point of interest (critical point) when $f_x = 0$ and $f_y = 0$

$f_x = 2x + y + 4$ $\qquad$ $2x + y + 4 = 0$

$f_y = x + 5$ $\qquad$ $x + 5 = 0$

solve $f_y = 0$ for x.

$x + 5 = 0$

$x = -5$

Sub. $x = -5$ in equation f_x and solve for y.

$2x + y + 4 = 0$

$2(-5) + y + 4 = 0$

$-10 + y + 4 = 0$

$y = 6$

Sub. $y = 6$ in equation for x.

N/A

thus, $(-5, 6)$ is a critical point.

2. Find D at $(a,b) = (-5,6)$

$f_x = 2x + y + 4$

$f_{xx} = 2$

$f_y = x + 5$

$f_{yy} = 0$

differenciate f_x with respect to y.

$f_{xy} = 2x + y + 6$

$= 1$

$f_{xx}f_{yy} - f_{xy^2} = 2 \cdot 0 - 1^2$

$= -1$

since $D < 0$, $f(-5,6) \rightarrow$ has saddle point.

14.7.6 Find the local maxima, local minima, and saddle points of the function.

$f(x,y) = x^2 - 4xy + y^2 + 6y + 4$

1. Find point of interest (critical point) when $f_x = 0$ and $f_y = 0$

$f_x = 2x - 4y$ $\qquad$ $2x - 4y = 0$

$f_y = -4x + 2y + 6$ $\qquad$ $-4x + 2y + 6 = 0$

solve $f_x = 0$ for x.

$2x - 4y = 0$

$2x = 4y$

$x = 2y$

Sub. $x = 2y$ in equation f_y and solve for y.

$-4x + 2y + 6 = 0$

$-4(2y) + 2y + 6 = 0$

$-8y + 2y + 6 = 0$

$-6y = -6$

$y = 1$

Sub. $y = 1$ in equation for x.

$x = 2y$

$x = 2(1)$

$x = 2$

thus, $(2,1)$ is a critical point.

2. Find D at $(a,b) = (2,1)$

$f_x = 2x - 4y$

$f_{xx} = 2$

$f_y = -4x + 2y + 6$

$f_{yy} = 2$

differenciate f_x with respect to y.

$f_{xy} = 2x - 4y$

$= -4$

$f_{xx}f_{yy} - f_{xy}{}^2 = (2)(2) - (-4)^2$

$= -12$

since $D < 0$, $f(2,1) \rightarrow$ has saddle point.

14.7.11 Find the local maxima, local minima, and saddle points of the function.

$$f(x,y) = \sqrt{12x^2 - 3y^2 - 8x - 15} + 1 - 4x$$

1. Find point of interest (critical point) when $f_x = 0$ and $f_y = 0$

Partial Differentiate f_x and f_y.

$$f_x = \frac{12x - 4}{\sqrt{12x^2 - 3y^2 - 8x - 15}} - 4$$

$$f_y = \frac{-3y}{\sqrt{12x^2 - 3y^2 - 8x - 15}}$$

Set $f_x = 0$ and $f_y = 0$

$$\frac{12x - 4}{\sqrt{12x^2 - 3y^2 - 8x - 15}} - 4 = 0$$

$$\frac{-3y}{\sqrt{12x^2 - 3y^2 - 8x - 15}} = 0$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$f(x) = \sqrt{12x^2 - 3y^2 - 8x - 15}$ $\qquad$ $g(x) = 12x^2 - 3y^2 - 8x - 15$

$f'(x) = \dfrac{1}{2\sqrt{12x^2 - 3y^2 - 8x - 15}}$ $\qquad$ $g'(x) = 24x - 8$

By the chain rule : $f'(g(x)) = f'(g(x))g'(x)$

$$\frac{1}{2\sqrt{12x^2 - 3y^2 - 8x - 15}}(24x - 8) = \frac{24\cancel{x - }8}{\cancel{2}\sqrt{12x^2 - 3y^2 - 8x - 15}} = \frac{12x - 4}{\sqrt{12x^2 - 3y^2 - 8x - 15}}$$

$$f_y = \frac{-\cancel{6}y}{\cancel{2}\sqrt{12x^2 - 3y^2 - 8x - 15}} = \frac{-3y}{\sqrt{12x^2 - 3y^2 - 8x - 15}}$$

Since f_x and f_y are not defined when the denominator $\sqrt{12x^2 - 3y^2 - 8x - 15} = 0$, find the function to which $f(x,y)$ simplifies when $\sqrt{12x^2 - 3y^2 - 8x - 15} = 0$

$$f(x,y) = 1 - 4x \text{ when } \sqrt{12x^2 - 3y^2 - 8x - 15} = 0$$

Find f_x and f_y for the simplified version of f on this set of points.

$$f_x = -4, \quad f_y = 0$$

Both f_x and f_y always exist and $f_x = -4$ is never equal to zero, $f(x,y)$ does not have any critical points satisfying $\sqrt{12x^2 - 3y^2 - 8x - 15} = 0$

Thus, only points where $f_x = 0$ and $f_y = 0$ need to be considered.

It can be assumed that:

$\sqrt{12x^2 - 3y^2 - 8x - 15} > 0$. The equation $\dfrac{-3y}{\sqrt{12x^2 - 3y^2 - 8x - 15}} = 0$ is satisfied for $y = 0$.

Therefore, any critical point of $f(x,y)$ has the form $(x, 0)$.

Substitute $y=0$ into f_x. Solve $f_x = 0$ for x.

$$\frac{12x - 4}{\sqrt{12x^2 - 3y^2 - 8x - 15}} - 4 = 0$$

$$\frac{12x - 4}{\sqrt{12x^2 - 3(0)^2 - 8x - 15}} - 4 = 0 \quad \text{substitute } y=0$$

$$\frac{12x - 4}{\sqrt{12x^2 - 8x - 15}} = 4$$

$$x = \frac{8}{3}$$

$$\sqrt{12x^2 - 8x - 15} \cdot \frac{12x - 4}{\sqrt{\cancel{12x^2 - 8x - 15}}} - 4 \cdot \sqrt{\cancel{12x^2 - 8x - 15}} = 0 \cdot \sqrt{12x^2 - 8x - 15}$$

$$12x - 4\sqrt{12x^2 - 8x - 15} - 4 = 0$$

$$-4\sqrt{12x^2 - 8x - 15} - 4 = -12x$$

$$\left(-4\sqrt{12x^2 - 8x - 15}\right)^2 = (-12x + 4)^2$$

$$(x+a)^2 = x^2 + 2ax + a^2$$

$$16(12x^2 - 8x - 15) = (-12x)^2 + 2(4)(-12x) + 4^2$$

$$192x^2 - 128x - 240 = 144x^2 - 96x + 16$$

$$192x^2 - 128x - 256 = 144x^2 - 96x$$

$$192x^2 - 32x - 256 = 144x^2 \qquad \frac{32}{48}\ \frac{16}{24}\ \frac{2}{3}$$

$$48x^2 - 32x - 256 = 0 \qquad \frac{256}{48}\ \frac{128}{24}\ \frac{16}{3}$$

$$x^2 - \frac{2}{3}x - \frac{16}{3} = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-\left(-\frac{2}{3}\right) \pm \sqrt{\left(-\frac{2}{3}\right)^2 - 4(1)\left(-\frac{16}{3}\right)}}{2(1)}$$

$$x = \frac{-\left(-\frac{2}{3}\right) \pm \frac{14}{3}}{2} = \frac{8}{3} \text{ and } -2$$

There is one critical point at $\left(\frac{8}{3}, 0\right)$

=

2. Find D at $\left(\frac{8}{3}, 0\right)$

$$f_x = \frac{12x-4}{\sqrt{12x^2-3y^2-8x-15}} - 4$$

$$f_{xx} = \frac{-36y^2-196}{(12x^2-3y^2-8x-15)^{3/2}}$$

$$f_y = \frac{-3y}{\sqrt{12x^2-3y^2-8x-15}}$$

$$f_{yy} = \frac{3(12x^2-8x-15)}{(12x^2-3y^2-8x-15)^{3/2}}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{\sqrt{12x^2-3y^2-8x-15}\cdot 12 - (12x-4)\frac{4(3x-1)}{\sqrt{12x^2-3y^2-8x-15}}}{\left(\sqrt{12x^2-3y^2-8x-15}\right)^2}$$

$$\frac{12\sqrt{12x^2-3y^2-8x-15} - \frac{4(3x-1)(12x-4)}{\sqrt{12x^2-3y^2-8x-15}}}{12x^2-3y^2-8x-15}$$

ANS from symbolab (stucked at "convert element to fraction)

differenciate f_x with respect to y.

$$f_{xy} = \frac{12x-4}{\sqrt{12x^2-3y^2-8x-15}} - 4$$
$$= \frac{3y(12x-4)}{(12x^2-3y^2-8x-15)^{3/2}}$$

evaluate f_{xx} at $\left(\frac{8}{3}, 0\right)$

$$f_{xx}\left(\frac{8}{3}, 0\right) = \frac{-36y^2-196}{(12x^2-3y^2-8x-15)^{3/2}}$$
$$= \frac{-36(0)^2-196}{\left(12\left(\frac{8}{3}\right)^2-3(0)^2-8\left(\frac{8}{3}\right)-15\right)^{3/2}} = \frac{-\cancel{196}}{\cancel{343}} = \boxed{-\frac{4}{7}}$$

evaluate f_{yy} at $\left(\frac{8}{3}, 0\right)$

$$f_{yy}\left(\frac{8}{3}, 0\right) = \frac{3(12x^2-8x-15)}{(12x^2-3y^2-8x-15)^{3/2}}$$
$$= \frac{3\left(12\left(\frac{8}{3}\right)^2-8\left(\frac{8}{3}\right)-15\right)}{\left(12\left(\frac{8}{3}\right)^2-3(0)^2-8\left(\frac{8}{3}\right)-15\right)^{3/2}} = \frac{\cancel{147}}{\cancel{343}} = \boxed{\frac{3}{7}}$$

evaluate f_{xy} at $\left(\frac{8}{3}, 0\right)$

$$f_{xy}\left(\frac{8}{3}, 0\right) = \frac{3y(12x-4)}{(12x^2-3y^2-8x-15)^{3/2}}$$
$$= \frac{3(0)\left(12\left(\frac{8}{3}\right)-4\right)}{\left(12\left(\frac{8}{3}\right)^2-3(0)^2-8\left(\frac{8}{3}\right)-15\right)^{3/2}} = \frac{0}{343} = \boxed{0}$$

evaluate D at $\left(\frac{8}{3}, 0\right)$

$$f_{xx}f_{yy} - f_{xy}^2 = -\frac{4}{7}\cdot\frac{3}{7} - 0^2 = \boxed{-\frac{12}{49}}$$

X

I DONT GIVE A FUCK

14.7.12 Find the local maxima, local minima, and saddle points of the function.

$$f(x,y) = 2 - \sqrt[3]{x^2 + y^2}$$

$$f_x = 2 - (x^2 + y^2)^{\frac{1}{3}}$$

$$= 0 - \frac{2x}{3\sqrt{x^2 + y^2}}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$= \frac{-2x}{3\sqrt[3]{x^2 + y^2}} \quad \text{or} \quad \frac{-2x}{3(x^2 + y^2)^{\frac{2}{3}}}$$

$$f_y = 2 - (x^2 + y^2)^{\frac{1}{3}}$$

$$= 0 - \frac{2y}{3\sqrt{x^2 + y^2}}$$

$$= \frac{-2y}{3\sqrt[3]{x^2 + y^2}} \quad \text{or} \quad \frac{-2y}{3(x^2 + y^2)^{\frac{2}{3}}}$$

Since $f_x = 0$ when $x = 0$ and $f_y = 0$ when $y = 0$.

Thus, the only critical point is $(0,0)$.

DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

the expression $\sqrt[3]{x^2 + y^2}$ is always ≥ 0.

therefore, the point $(0,0)$ is a local maximum.

substitute the point $(0,0)$ into the original function.

$$f(x,y) = 2 - \sqrt[3]{x^2 + y^2}$$

$$f(0,0) = 2 - \sqrt[3]{0^2 + 0^2}$$

$$= 2$$

thus, the function has a local maximum with value of 2 at $(0,0)$.

14.7.15 Find the local maxima, local minima, and saddle points of the function.

$f(x,y) = 10x^2 - 2x^3 + y^2 + 2xy$

$f_x = 20x - 6x^2 + 2y$

$f_y = 2y + 2x$

solve $f_y = 0$ for x.

$2y + 2x = 0$

$2x = -2y$

$x = -y$

substitute $x = -y$ in the equation f_x and solve for y.

$20x - 6x^2 + 2y = 0$

$20(-y) - 6(-y)^2 + 2y = 0$

$-20y - 6y^2 + 2y = 0$

$-6y^2 - 18y = 0$

$y^2 + 3y = 0$

$y(y+3) = 0$

$y = 0, -3$

substitute $y=0$ in the equation for x.

$2(0) + 2x = 0$

$2x = 0$

$x = 0$

substitute $y=-3$ in the equation for x.

$2(-3) + 2x = 0$

$-6 + 2x = 0$

$2x = 6$

$x = 3$

Hence, the critical points of the function f

are: $(0,0)$ and $(3,-3)$

Find discriminant $f_{xx}f_{yy} - f_{xy}^2$

$f_x = 20x - 6x^2 + 2y$

$f_{xx} = 20 - 12x$

$f_y = 2y + 2x$

$f_{yy} = 2$

differenciate f_x with respect to y.

$f_{xy} = 20x - 6x^2 + 2y$

$= 2$

evaluate the discriminant at $(0,0)$

$f_{xx}f_{yy} - f_{xy}^2 = 20 - 12x \cdot 2 - 2^2$

$= 40 - 24(0) - 4$

$= 36$

since the discriminant > 0 and $f_{xx} > 0$,

$(0,0)$ is a local minimum.

evaluate the discriminant at $(3,-3)$

$f_{xx}f_{yy} - f_{xy}^2 = 20 - 12x \cdot 2 - 2^2$

$= 40 - 24(3) - 4$

$= -36$

since the discriminant < 0,

$(3,-3)$ is a saddle point.

Find the value of the local minimum at $(0,0)$

$f(x,y) = 10x^2 - 2x^3 + y^2 + 2xy$

$= 10(0)^2 - 2(0)^3 + 0^2 + 2(0)(0)$

$= 0$

14.7.16 Find the local maxima, local minima, and saddle points of the function.

$f(x,y) = x^3 + y^3 + 3x^2 - 9y^2 - 2$

$f_x = 3x^2 + 6x$

$f_y = 3y^2 - 18y$

Find critical points, set $f_x = 0$ and $f_y = 0$

$3x^2 + 6x = 0$ $\qquad$ $3y^2 - 18y = 0$

$3x(x+2) = 0$ $\qquad$ $3y(y-6) = 0$

$x = 0, -\cancel{3}-2$ $\qquad$ $y = 0, 6$

Hence, the critical points of the function f are: $(0,0)$, $(0,6)$, $(\overset{-2}{-\cancel{3}},0)$ and $(\overset{-2}{-\cancel{3}},6)$.

Find discriminant $f_{xx}f_{yy} - f_{xy^2}$

$f_x = 3x^2 + 6x$

$f_{xx} = 6x + 6$

$f_y = 3y^2 - 18y$

$f_{yy} = 6y - 18$

$f_{xy} = 3x^2 + 6x$

$= 0$

evaluate the discriminant at $(0,0)$

$$f_{xx}f_{yy} - f_{xy^2} = (6\overset{0}{x}+6)(6\overset{0}{y}-18) - 0^2$$
$$= (6)(-18)$$
$$= -108$$

since the discriminant < 0, $(0,0)$ is a saddle point.

evaluate the discriminant at $(0,6)$

$$f_{xx}f_{yy} - f_{xy^2} = (6\overset{0}{x}+6)(6\overset{6}{y}-18) - 0^2$$
$$= (6)(18)$$
$$= 108$$

since the discriminant > 0 and $F_{xx} > 0$, $(0,6)$ is a local minimum.

Find the value of the local minimum at $(0,6)$

$$f(x,y) = \overset{0}{x}^3 + \overset{6}{y}^3 + 3x^2 - 9\overset{6}{y}^2 - 2$$
$$= 216 - 324 - 2$$
$$= -110$$

Cont.

evaluate the discriminant at $(\overset{-2}{-\cancel{3}},0)$

$$f_{xx}f_{yy} - f_{xy^2} = (6\overset{-2}{x}+6)(6\overset{0}{y}-18) - 0^2$$
$$= -6(-18)$$
$$= 108$$

since the discriminant > 0 and $F_{xx} < 0$, $(\overset{-2}{-\cancel{3}},0)$ is a local maximum.

Find the value of the local maximum at $(\overset{-2}{-\cancel{3}},0)$

$$f(x,y) = \overset{-2}{x}^3 + \overset{0}{y}^3 + 3\overset{-2}{x}^2 - 9\overset{0}{y}^2 - 2$$
$$= -8 + 12 - 2$$
$$= 2$$

evaluate the discriminant at $(\overset{-2}{-\cancel{3}},6)$

$$f_{xx}f_{yy} - f_{xy^2} = (6\overset{-2}{x}+6)(6\overset{6}{y}-18) - 0^2$$
$$= -6(18)$$
$$= -108$$

since the discriminant < 0, $(\overset{-2}{-\cancel{3}},6)$ is a saddle point.

therefore: the saddle points are $(0,0)$ and $(-2,6)$

the local maximum is $(-2,0)$ with value of 2.

the local minimum is $(0,6)$ with value of -110.

14.7.21 Find the local maxima, local minima, and saddle points of the function.

$$f(x,y) = \frac{6}{x^2+y^2-1}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{(x^2+y^2-1)(0) - (6)(2x)}{(x^2+y^2-1)^2} = \frac{-12x}{(x^2+y^2-1)^2}$$

$$\frac{(x^2+y^2-1)(0) - (6)(2y)}{(x^2+y^2-1)^2} = \frac{-12y}{(x^2+y^2-1)^2}$$

$$f_x = \frac{-12x}{(x^2+y^2-1)^2} \qquad f_y = \frac{-12y}{(x^2+y^2-1)^2}$$

Find critical points, set $f_x = 0$ and $f_y = 0$

$$\frac{-12x}{(x^2+y^2-1)^2} = 0 \qquad \frac{-12y}{(x^2+y^2-1)^2} = 0$$

$$x = 0 \qquad y = 0$$

Hence, the critical point of the function f is $(0,0)$

$$f_x = \frac{-12x}{(x^2+y^2-1)^2} \qquad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \qquad f_y = \frac{-12y}{(x^2+y^2-1)^2}$$

$$f_{xx} = -12 \cdot \frac{(x^2+y^2-1)^2(1) - (x)(2(x^2+y^2-1))(2x)}{((x^2+y^2-1)^2)^2}$$

$$= -12 \cdot \frac{(x^2+y^2-1)^2 - (x)(4x(x^2+y^2-1))}{((x^2+y^2-1)^2)^2}$$

$$= -12 \cdot \frac{(x^2+y^2-1)(x^2+y^2-1) - 4x^2(x^2+y^2-1)}{(x^2+y^2-1)^4}$$

$$= -12 \cdot \frac{\cancel{(x^2+y^2-1)}(x^2+y^2-1-4x^2)}{(x^2+y^2-1)^{\cancel{4}}}$$

$$= \frac{-12(-3x^2+y^2-1)}{(x^2+y^2-1)^3}$$

$$f_{yy} = -12 \cdot \frac{(x^2+y^2-1)^2(1) - (y)(2(x^2+y^2-1))(2y)}{((x^2+y^2-1)^2)^2}$$

$$= -12 \cdot \frac{(x^2+y^2-1)^2 - (y)(4y(x^2+y^2-1))}{((x^2+y^2-1)^2)^2}$$

$$= -12 \cdot \frac{(x^2+y^2-1)(x^2+y^2-1) - 4y^2(x^2+y^2-1)}{(x^2+y^2-1)^4}$$

$$= -12 \cdot \frac{\cancel{(x^2+y^2-1)}(x^2+y^2-1-4y^2)}{(x^2+y^2-1)^{\cancel{4}}}$$

$$= \frac{-12(x^2-3y^2-1)}{(x^2+y^2-1)^3}$$

$$f_{xy} = \frac{-12x}{(x^2+y^2-1)^2}$$

$$= -12x \cdot \frac{\cancel{(x^2+y^2-1)^2(0)} - (0)(2\cancel{(x^2+y^2-1)})(2y)}{((x^2+y^2-1)^2)^{\cancel{2}}}$$

$$= \frac{-12x \cdot 2(2y)}{(x^2+y^2-1)^3}$$

$$= \frac{-48xy}{(x^2+y^2-1)^3}$$

evaluate f_{xx} at $(0,0)$

$$f_{xx}(0,0) = \frac{-12(-3x^2+y^2-1)}{(x^2+y^2-1)^3} = -12$$

evaluate f_{yy} at $(0,0)$

$$f_{yy}(0,0) = \frac{-12(x^2-3y^2-1)}{(x^2+y^2-1)^3} = -12$$

evaluate f_{xy} at $(0,0)$

$$f_{xy}(0,0) = \frac{-48xy}{(x^2+y^2-1)^3} = 0$$

evaluate the discriminant

$$f_{xx}f_{yy} - f_{xy}^2 = -12 \cdot -12 - 0^2 = 144$$

Find the value of the local maximum at $(0,0)$

$$f(x,y) = \frac{6}{x^2+y^2-1} = -6$$

since the discriminant > 0 and $f_{xx} < 0$, $(0,0)$ is a local maximum with value of -6.

14.7.23 Find the local maxima, local minima, and saddle points of the function.

$f(x,y) = 5y\sin(x)$

①

Q $f(x,y) = 5y\sin x$

$f_x = 5y\cos x$; $f_y = 5\sin x$

$f_{xx} = -5y\sin x$; $f_{yy} = 0$

$f_{xy} = 5\cos x$, $\left\{ \text{Here } f_x = \frac{\partial}{\partial x} f(x,y) ;\ f_y = \frac{\partial}{\partial y}(f(x,y)) ;\ f_{xy} = \frac{\partial}{\partial y}(f_x) ,\ f_{yy} = \frac{\partial}{\partial y}(f_y) \right\}$

⇒ To find the critical points —

$f_x = 0$; $f_y = 0$

$5y\cos x = 0$; $5\sin x = 0$

The factor $\cos x$ will never zero for the value of $x = n\pi$ ⇒ $\sin x = 0$

$x = n\pi$; where n is an integer

so ;

$y = 0$

the critical points are → $P_c = (n\pi, 0)$

Now ; $D = f_{xx} f_{yy} - (f_{xy})^2$

$= 5y\cos x \times 0 - (5\cos x)^2$

$D = 0 - (5\cos x)^2 = -25(\cos x)^2$

$D(a,b)$ or the value of D at critical points

$D = -25(\cos n\pi)^2$.

②

cosine of $n\pi$ is $+1$ for even integers n

& -1 for odd integers n.

But we square that so we always are left with a $+1$ for cosine squared.

So $D = -25(+1)^2 = -25(-1)^2 = -25$

$D = -25$

We also need; $f_{xx} = \cancel{5\cos x}\ -5y\sin x$

It doesn't matter what is x ; we require that $y = 0$

Therefore $f_{xx} = 0$ for all critical points.

Now the general list of the conditions for the test.

i) Local minima : $D > 0$ and $f_{xx} > 0$

ii) Local maxima : $D > 0$ and $f_{xx} < 0$

iii) saddle point : $D < 0$

So for the function

Ans (i) There is no local minima

(ii) There is no local maxima

(iii) All the critical points are saddle points.

& saddle points are located at $(n\pi, 0)$; where n is an integer

14.7.26 Find the local maxima, local minima, and saddle points of the function.

$f(x,y) = 7e^y - 6ye^x$

$f(x,y) = 9e^y - 7ye^x$ Print

The local extrema and saddle points in the interior of the domain of a function occur at critical points of the function. A point is a critical point if both f_x and f_y are zero or if one or both derivatives do not exist.

Since the domain of $f(x,y) = 9e^y - 7ye^x$ is the entire plane, the domain of f does not have any boundary points.

To find the critical points of f, first find f_x.

$f_x = -7ye^x$ $\quad -6ye^x$

Now find f_y.

$f_y = 9e^y - 7e^x$ $\quad 7e^y - 6e^x$

Note that there are no values of x or y for which f_x or f_y do not exist. Since there are no boundary points, the function can have extreme values and saddle points only at critical points where f_x and f_y are zero or where one or both derivatives do not exist.

Proceed by letting $f_x = -7ye^x = 0$. Notice that no value of x will make $-7ye^x = 0$ a true statement. The value of y that satisfies this equation is $y = 0$.

Therefore, any critical point of f must have y-coordinate 0. Determine the x-coordinate(s).

$9e^{y} - 7e^x = 9 - 7e^x$ Let $f_y = 0$. $\quad 7e^y - 6e^x = 0$ $\quad$ Cont. $-6e^x = -7$

$9 - 7e^x = 0$ $\quad \frac{-7}{-6}$ Let $y = 0$. $\quad 7e^0 - 6e^x = 0$ $\quad e^x = \frac{-7}{-6}$

$x = \ln\frac{9}{7}$ Solve for x. $\quad 7 - 6e^x = 0$ $\quad e^x = \frac{7}{6} = \ln\frac{7}{6}$

$e^x = \frac{7}{6}$

Thus, the only critical point of f is $\left(\ln\frac{9}{7}, 0\right)$. $\left(\ln\frac{7}{6}, 0\right)$

Now that the critical point of f is known, use the Second Derivative Test to determine whether the point is a local minimum, a local maximum, or a saddle point.

In order to apply Second Derivative Test, find the discriminant of f, which is given by $f_{xx}f_{yy} - f_{xy}^2$. Calculate the second partial derivatives, starting with f_{xx}.

$f_x = -7ye^x$ $\quad -6ye^x$

$f_{xx} = -7ye^x$ $\quad -6ye^x$

Next calculate f_{yy}.

$f_y = 9e^y - 7e^x$ $\quad 7e^y - 6e^x$

$f_{yy} = 9e^y$ $\quad 7e^y$

Calculate f_{xy}.

$f_x = -7ye^x$ $\quad -6ye^x$

$f_{xy} = -7e^x$ $\quad -6e^x$

Now calculate $f_{xx}f_{yy} - f_{xy}^2$, where $f_{xx} = -7ye^x$, $f_{yy} = 9e^y$, and $f_{xy} = -7e^x$.

$f_{xx}f_{yy} - f_{xy}^2 = -63ye^{x+y} - 49e^{2x}$ $\quad -6ye^x \cdot 7e^y - (-6e^x)^2 = -42ye^{x+y} - 36e^{2x}$

The nature of the critical point can be tested using the Second Derivative Test if the first- and second-order partial derivatives of f are continuous throughout a disk centered at point (a,b) and $f_x(a,b) = f_y(a,b) = 0$.

i. If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b), then f has a local maximum at (a,b).
ii. If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b), then f has a local minimum at (a,b).
iii. If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b), then f has a saddle point at (a,b).
iv. If $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b), then the test is inconclusive at (a,b).

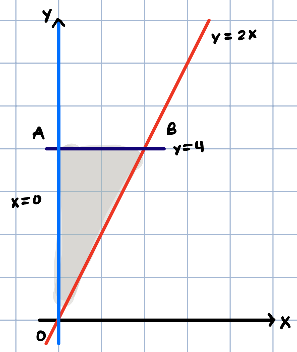
Apply the Second Derivative Test to the critical point $\left(\ln\frac{7}{6}, 0\right)$. Determine symbols below that make the statements true. Use $f_{xx}f_{yy} - f_{xy}^2 = -63ye^{x+y} - 49e^{2x}$ and $f_{xx} = -7ye^x$.

$f_{xx}f_{yy} - f_{xy}^2 < 0$ and $f_{xx} = 0$

Since $f_{xx}f_{yy} - f_{xy}^2 < 0$ and $f_{xx} = 0$, the function f(x,y) has a saddle point at the critical point $\left(\ln\frac{7}{6}, 0\right)$.

Therefore, the function $f(x,y) = 9e^y - 7ye^x$ has a saddle point at $\left(\ln\frac{9}{7}, 0\right)$ and no local maxima or minima.

14.7.31 Find the absolute maximum and minimum of the function $f(x,y) = 2x^2 - 8x + y^2 - 8y + 2$ on the closed triangular plate bounded by the lines $x = 0$, $y = 4$, and $y = 2x$ in the first quadrant.



$f_x = 4x - 8$

$4x - 8 = 0$

$x = 2$

$f_y = 2y - 8$

$2y - 8 = 0$

$y = 4$

the point $(2,4)$ is on the boundary of R, not on the interior.

therefore, it cannot be considered yet.

Now, consider the boundary of R one segment at a time.

On the segment OA, $x = 0$

therefore,

$f(x,y) = 2x^2 - 8x + y^2 - 8y + 2$

$f(0,y) = y^2 - 8y + 2$

compute the values of $f(0,y)$ at the endpoints $(0,0)$ and $(0,4)$.

$f(0,y) = y^2 - 8y + 2$

$f(0,0) = 2$

$f(0,y) = y^2 - 8y + 2$

$f(0,4) = -14$

Now find $f'(0,y)$

$f(0,y) = y^2 - 8y + 2$

$f'(0,y) = 2y - 8$

set $f'(0,y) = 0$

$2y - 8 = 0$

$y = 4$

$y=4$ corresponds to one of the end points of the segment OA. As computed above $f(0,4) = -14$.

On the segment AB, $y = 4$

therefore,

$f(x,y) = 2x^2 - 8x + y^2 - 8y + 2$

$f(x,4) = 2x^2 - 8x - 14$

compute the values of $f(x,4)$ at the endpoints $(0,4)$ and $(2,4)$.

$f(x,4) = 2x^2 - 8x - 14$

$f(0,4) = -14$

$f(x,4) = 2x^2 - 8x - 14$

$f(2,4) = -22$

Now find $f'(x,4)$

$f(x,4) = 2x^2 - 8x - 14$

$f'(x,4) = 4x - 8$

set $f'(0,y) = 0$

$4x - 8 = 0$

$x = 2$

$x=2$ corresponds to one of the end points of the segment AB. As computed above $f(2,4) = -22$.

On the segment OB, $y = 2x$

therefore,

$f(x,y) = 2x^2 - 8x + y^2 - 8y + 2$

$f(x,2x) = 2x^2 - 8x + 4x^2 - 16x + 2$

$f(x,2x) = 6x^2 - 24x + 2$

compute the values of $f(x,2x)$ at the endpoints $(0,0)$ and $(2,4)$.

$f(x,2x) = 6x^2 - 24x + 2$

$f(0,0) = 2$

$f(x,2x) = 6x^2 - 24x + 2$

$f(2,4) = -22$

Now find $f'(x,2x)$

$f(x,2x) = 6x^2 - 24x + 2$

$f'(x,2x) = 12x - 24$

set $f'(x,2x) = 0$

$12x - 24 = 0$

$x = 2$

$x=2$ corresponds to one of the end points of the segment OB. As computed above $f(2,4) = -22$.

therefore, the choices for absolute maxima and minima of f over the given domain are as follows:

$f(0,0) = 2$ absolute maximum

$f(0,4) = -14$

$f(2,4) = -22$ absolute minimum

14.7.35 Find the absolute maxima and minima of the function on the given domain.
$T(x,y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5, -3 \le y \le 0$.

FOUND ANS ON CHEGG.

14.8.1 Find the points on the ellipse $3x^2 + y^2 = 1$ where $f(x,y) = xy$ has its extreme values.

Find the values of x, y, and λ for which $\nabla f = \lambda \nabla g$ and $g(x,y) = 0$.

$\nabla f = yi + xj$

find the gradient of $g(x,y) = 3x^2 + y^2 - 1 = 0$

$\lambda \nabla g = 6\lambda x i + 2\lambda y j$

therefore, any points that are extreme values of $f(x,y) = xy$ given the constraint $3x^2 + y^2 - 1 = 0$, must satisfy the equation $yi + xj = 6\lambda x i + 2\lambda y j$.

thus, the coordinates of any such extreme values must satisfy the scalar equations $y = 6\lambda x$ and $x = 2\lambda y$

substitute the expression for x into the expression for y and solve for λ.

$y = 6\lambda x$
$= 6\lambda(2\lambda y)$
$= 12\lambda^2 y$

$x = 2\lambda y$
$= 2\lambda(6\lambda x)$
$=$

one sol is $y=0$ in which case $x=0$. But $(0,0)$ does not exist in the ellipse. so, $y \neq 0$.

solve for $y \neq 0$

$y = 12\lambda^2 y$

$\sqrt{\frac{1}{12}} = \sqrt{\lambda^2}$

$\pm \frac{1}{2\sqrt{3}} = \lambda$

substitute the value of λ and solve for x.

$x = 2\lambda y$

$x = 2\left(\pm \frac{1}{2\sqrt{3}}\right) y$

$x = \pm \frac{2}{\sqrt{3}} y$

substitute the expression for x into the original equation and solve for y.

$3x^2 + y^2 = 1$

$3\left(\pm \frac{2}{\sqrt{3}} y\right)^2 + y^2 = 1$

$4y^2 + y^2 = 1$

$5y^2 = 1$

$y^2 = \frac{1}{5}$

$y = \pm \frac{1}{\sqrt{5}}$

therefore, the y-coordinates of the extrema are $y = \pm \frac{1}{\sqrt{5}}$.

cont. to find the x-coordinates, substitute y into $x = \frac{2}{\sqrt{3}} y$

$x = \frac{2}{\sqrt{3}} y$

$x = \frac{2}{\sqrt{3}} \left(\pm \frac{1}{\sqrt{5}}\right)$

$x = \pm \frac{2}{\sqrt{15}}$

thus, if the y-coordinates are $y = \pm \frac{1}{\sqrt{5}}$ and the x-coordinates are $x = \pm \frac{2}{\sqrt{15}}$, $f(x,y) = xy$ takes on its extreme values on the ellipse at the four points $\left(\frac{2}{\sqrt{15}}, \frac{1}{\sqrt{5}}\right), \left(-\frac{2}{\sqrt{15}}, \frac{1}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{15}}, \frac{-1}{\sqrt{5}}\right), \left(\frac{-2}{\sqrt{15}}, \frac{-1}{\sqrt{5}}\right)$.

$3x^2+y^2=1$, $f(x,y)=xy$

let,

$g(x,y) = 3x^2+y^2=1$

By lagrange multiplers

$\nabla f = \lambda \nabla g$

$\langle y, x\rangle = \lambda \langle 6x, 2y\rangle$

$6x\lambda = y$ —(1) , $x = 2\lambda y$ —(2)

Substitue (2) in (1) we get

$x = 2\lambda(6x\lambda) \Rightarrow x = 12x\lambda^2$, $x = 0$

$\lambda^2 = 1/12$

$\lambda = \pm\frac{1}{\sqrt{12}} = \frac{\pm 1}{2\sqrt{3}}$

Substitute , $x=0$, then g yields

$y^2=1$ $\therefore$ $y=\pm 1$

If $\lambda = \pm 1/2\sqrt{3}$, then $y = 6x\left(\pm\frac{1}{2\sqrt{3}}\right) = \pm\frac{3x}{\sqrt{3}}$

$y = \pm x\sqrt{3}$

Substituting in g

$3x^2 + x^2(3) = 1 \Rightarrow 6x^2 = 1$

$x^2 = 1/6 \Rightarrow x = \pm 1/\sqrt{6}$

$\therefore f(0, \pm 1/2\sqrt{3}) = 0$ $y = \pm 1/\sqrt{2}$

$f\left(\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{2}}\right) = 1/2\sqrt{3}$ [maximum]

$f(\pm 1/\sqrt{6}, \mp 1/\sqrt{2}) = -\frac{1}{2\sqrt{3}}$ [minimum]

14.8.2 Find the extreme values of $f(x,y) = xy$ subject to the constraint $x^2+y^2-10=0$.

Find the values of x, y, and λ for which $\nabla F = \lambda \nabla g$ and $g(x,y)=0$.

$\nabla F = yi + xj$

Find the gradient of $g(x,y) = x^2+y^2-10=0$.

$\lambda \nabla g = 2\lambda x i + 2\lambda y j$

Substitute the value of $x = 2\lambda y$ into $y = 2\lambda x$

$y = 2\lambda x$

$y = 2\lambda(2\lambda y)$

$y = 4\lambda^2 y$

Solve for $y \neq 0$

$y = 4\lambda^2 y$

$\frac{1}{4} = \lambda^2$

$\lambda = \pm\frac{1}{2}$

use $\lambda = \pm\frac{1}{2}$ in $x = 2\lambda y$

$x = 2\lambda y$

$x = \cancel{2}(\pm\frac{1}{\cancel{2}})y$

$x = \pm y$

Substitute the value of $x = \pm y$ into $g(x,y) = 0$.

$x^2+y^2=10$ $\rightarrow$ $y^2 = 5$

$(\pm y)^2 + y^2 = 10$ $y = \pm\sqrt{5}$ y-coordinates of the extrema

$y^2+y^2=10$

$2y^2 = 10$

Substitute $y = \pm\sqrt{5}$ into $x = \pm y$ to find the x-coordinates

$x = \pm y$

$x = \pm(\pm\sqrt{5})$

$x = \pm\sqrt{5}$ x-coordinates of the extrema

therefore, any points that are extreme values of $f(x,y)=xy$ given the constraint $x^2+y^2-10=0$ must satisfy the equation $yi + xj = 2\lambda x i + 2\lambda y j$.

thus, the coordinates of any such extreme values must satisfy the scalar equations $y = 2\lambda x$ and $x = 2\lambda y$

cont. determine the values of the function by substituting x and y into $f(x,y) = xy$

$f(x,y) = xy$

$f(x,y) = (\pm\sqrt{5})(\pm\sqrt{5})$

$f(x,y) = \pm 5$

thus, the extreme values of $f(x,y) = xy$ are 5 and -5.

14.8.9 Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is 1024π cm^3

r = radius

h = height

the equation for the surface area of a right circular cylindrical can:

$$SA = 2\pi r^2 + 2\pi rh$$

the equation for the volume of a right circular cylindrical can:

$$V = \pi r^2 h$$

It is given that the volume is 1024π cm^3, $V = \pi r^2 h = 1024\pi$

suppose $f(r,h)$ and $g(r,h)$ are differentiable. To find the local minimum of f subject to the constraint of $g(r,h) = 0$, find the values of r, h, and λ that simultaneously satisfy the equations $\nabla f = \lambda \nabla g$ and $g(r,h) = 0$.

in this case, $f(r,h) = 2\pi r^2 + 2\pi rh$ and $g(r,h) = \pi r^2 h - 1024\pi$

find ∇f

$f(r,h) = 2\pi r^2 + 2\pi rh$

$\nabla f = (4\pi r + 2\pi h)i + (2\pi r)j$

find ∇g

$g(r,h) = \pi r^2 h - 1024\pi$

∇g

apply the Lagrange method in order to find the values of r, h, and λ.

$\nabla f = \lambda \nabla g$

$(4\pi r + 2\pi h)i + (2\pi r)j = \lambda(2\pi rh\, i + \pi r^2 j)$

use the equation $2\pi r = \lambda \pi r^2$ Note: that this is true when $r = 0$, but $r \neq 0$ if the volume is 1024π.

so, use $2\pi r = \lambda \pi r^2$ to solve for λ.

$2\pi r = \lambda \pi r^2$

$\dfrac{2\cancel{\pi r}}{\cancel{\pi r^2}} = \lambda$

$\lambda = \dfrac{2}{r}$

find h in terms of r using the fact that $4\pi r + 2\pi h = \lambda 2\pi rh$ and $\lambda = \dfrac{2}{r}$

$4\pi r + 2\pi h = \lambda 2\pi rh$

$4\pi r + 2\pi h = \left(\dfrac{2}{\cancel{r}}\right) 2\pi \cancel{r} h$

$4\pi r + 2\pi h = 4\pi h$

$4\pi r = 2\pi h$

$h = 2r$

Now substitute $h = 2r$ into the equation $\pi r^2 h = 1024\pi$ to find r.

$\pi r^2 h = 1024\pi$

$\pi r^2 (2r) = 1024\pi$

$2\pi r^3 - 1024\pi = 0$

$2\pi(r^3 - 512) = 0$ Factor 2π

$2\pi(r^3 - 8^3) = 0$ Factor $(r^3 - 512)$ rewrite 512 as 8^3

$2\pi(r - 8)(r^2 + 8r + 8^2) = 0$ $\quad x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$r = 8$

use the equation $h = 2r$ to find h.

$h = 2r$

$h = 2(8)$

$h = 16$

thus, the closed right circular cylindrical can of smallest surface area whose volume is 1024π cm^3 has a radius of 8 cm and a height of 16 cm.

14.8.18 find the point on the sphere $x^2+y^2+z^2=9$ farthest from the point $(-1,-1,-1)$.
the distance between the point (x,y,z) on the surface $g(x,y,z)=x^2+y^2+z^2-9$
and the point $(-1,-1,-1)$ is given by a function $f(x,y,z)=(x-(-1))^2+(y-(-1))^2+(z-(-1))^2$

to maximize f subject to the constraint g:

find $\nabla f = f_x i + f_y j + f_z k$

$$\nabla f = f_x\left[(x-(-1))^2+(y-(-1))^2+(z-(-1))^2\right]i$$
$$+ f_y\left[(x-(-1))^2+(y-(-1))^2+(z-(-1))^2\right]j$$
$$+ f_z\left[(x-(-1))^2+(y-(-1))^2+(z-(-1))^2\right]k$$

$$\nabla f = 2(x+1)i + 2(y+1)j + 2(z+1)k$$

find $\nabla g = g_x i + g_y j + g_z k$

$$\nabla g = g_x(x^2+y^2+z^2-9)i + g_y(x^2+y^2+z^2-9)j + g_z(x^2+y^2+z^2-9)k$$
$$\nabla g = 2xi + 2yj + 2zk$$

find $\nabla f = \lambda \nabla g$

$$2(x+1)i + 2(y+1)j + 2(z+1)k = 2xi + 2yj + 2zk$$

thus, $\cancel{2}(x+1) = \lambda\cancel{2}x$ $\quad \cancel{2}(y+1) = \lambda\cancel{2}y$ $\quad \cancel{2}(z+1) = \lambda\cancel{2}z$

$(x+1) = \lambda x$ $\quad (y+1) = \lambda y$ $\quad (z+1) = \lambda z$

$x+1 = \lambda x$ $\quad y+1 = \lambda y$ $\quad z+1 = \lambda z$

$x = \lambda x - 1$ $\quad y = \lambda y - 1$ $\quad z = \lambda z - 1$

factor $\quad x - \lambda x = -1$ $\quad y - \lambda y = -1$ $\quad z - \lambda z = -1$

$x(1-\lambda) = -1$ $\quad y(1-\lambda) = -1$ $\quad z(1-\lambda) = -1$

$x = \dfrac{-1}{1-\lambda}$ $\quad y = \dfrac{-1}{1-\lambda}$ $\quad z = \dfrac{-1}{1-\lambda}$ $\quad \lambda \neq 1$

substitute these values into the equation $x^2+y^2+z^2=9$

$$\left(\frac{-1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 = 9$$

$$\frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 9$$

$$1+1+1 = 9(1-\lambda)^2$$
$$3 = 9(1-\lambda)^2$$
$$\frac{3}{9} = (1-\lambda)^2$$
$$\sqrt{\frac{1}{3}} = \sqrt{(1-\lambda)^2}$$
$$1-\lambda = \sqrt{\frac{1}{3}}$$
$$\frac{-\lambda}{-1} = \frac{\sqrt{\frac{1}{3}}}{-1} \frac{-1}{-1}$$
$$\lambda = -\sqrt{\frac{1}{3}} \cancel{+1}$$

thus, $\lambda = \pm \dfrac{\cancel{1}}{\sqrt{3}} \cancel{+1}$

Find the largest product the positive numbers x, y, and z can have if $x^2 + y + z = 25$.

The product of three positive numbers is xyz.

Assume $f(x,y,z) = xyz$.

Let $g(x,y,z) = x + y + z^2 - 16 = 0$. $x^2 + y + z - 25 = 0$

Use Lagrange multiplier method, to find the largest product as follows:

$$\nabla f = \lambda \nabla g.$$

The gradient of the functions f and g are calculated as follows:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$
$$= \mathbf{i}\frac{\partial}{\partial x}(xyz) + \mathbf{j}\frac{\partial}{\partial y}(xyz) + \mathbf{k}\frac{\partial}{\partial z}(xyz)$$
$$= \mathbf{i}yz\frac{\partial}{\partial x}(x) + \mathbf{j}xz\frac{\partial}{\partial y}(y) + \mathbf{k}xy\frac{\partial}{\partial z}(z)$$
$$= (yz)\mathbf{i} + (xz)\mathbf{j} + (xy)\mathbf{k}$$

$$\nabla g = \mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z}$$
$$= \mathbf{i}\frac{\partial}{\partial x}\left(\overset{x^2+y+z-25}{x + y + z^2 - 16}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(\overset{x^2+y+z-25}{x + y + z^2 - 16}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(\overset{x^2+y+z-25}{x + y + z^2 - 16}\right)$$
$$= \mathbf{i}\cdot 1 + \mathbf{j}\cdot 1 + \mathbf{k}(2z)$$
$$= \mathbf{i} + \mathbf{j} + (2z)\mathbf{k} \quad 2x\,\mathbf{i} + \mathbf{j} + \mathbf{k}$$

Plug these values into $\nabla f = \lambda \nabla g$.

$$(yz)\mathbf{i} + (xz)\mathbf{j} + (xy)\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + (2z)\mathbf{k}).\ \lambda(2x\,\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Equate the like terms on both sides, then

$yz = \lambda, xz = \lambda, xy = \lambda(2z)$. $yz = \lambda(2x)$, $xz = \lambda$, $xy = \lambda$

This can be written as,

$yz = xz$ $\quad xz = xy$

$y = x$ $\quad z = y$

And,

$xy = \lambda(2z)$ $\quad yz = \lambda(2x)$

$\lambda = \frac{xy}{2z}$ $\quad \lambda = \frac{yz}{2x}$

$yz = \frac{xy}{2z}$ Use $yz = \lambda$ $\quad xz = \frac{yz}{2x}$ use $xz = \lambda$

$x = 2z^2$ $\quad y = 2x^2$

$z^2 = \frac{x}{2}$ $\quad x^2 = \frac{y}{2}$ $\quad x^2 = \frac{y}{2}$

Plug ~~x = y~~ $z = y$ and ~~$z^2 = \frac{x}{2}$~~ $x^2 = \frac{y}{2}$ into ~~$x + y + z^2 - 16$~~ $= 0$. $x^2 + y + z - 25$

$x + x + \frac{x}{2} - 16 = 0$ Use $y = x, z^2 = \frac{x}{2}$

$\frac{5x}{2} = 16$ $\quad \frac{y}{2} + y + y - 25 = 0$

$5x = 32$ $\quad \frac{y}{2} + 2y = 25$

$x = \frac{32}{5}$ $\quad 2 \cdot \frac{y}{2} + 2y \cdot 2 = 25 \cdot 2$

That is,

$x = y = \frac{32}{5}$. $\quad y + 4y = 50$

$y = z = 10$

And, $\quad 5y = 50$

$z^2 = \frac{\frac{32}{5}}{2}$ $\quad x^2 = \frac{10}{2}$ $\quad y = 10$

$= \frac{32}{10}$ $= 5$

$\sqrt{x}\,x\,z = \sqrt{\frac{16}{5}}$ $= \sqrt{5}$ $\quad = (\sqrt{5})(10)(10)$

$x = \frac{4}{\sqrt{5}} = \sqrt{5}$

$= \frac{4\sqrt{5}}{5}$ $\quad = 100\sqrt{5}$

So, multiply these three numbers to get largest number.

$$f\left(\frac{32}{5}, \frac{32}{5}, \frac{4\sqrt{5}}{5}\right) = \left(\frac{32}{5}\right)\left(\frac{32}{5}\right)\left(\frac{4\sqrt{5}}{5}\right)$$
$$= \frac{4096\sqrt{5}}{125}$$

A space probe in the shape of the ellipsoid $9x^2 + y^2 + 4z^2 = 20$ enters a planet's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x,y,z) on the probe's surface is $T(x,y,z) = 18x^2 + 4yz - 16z + 597$. Find the hottest point on the probe's surface.

to find the hottest point on the probe's surface where $T = F = 18x^2 + 4yz - 16z + 597$

with the constraint $g = 9x^2 + y^2 + 4z^2 = 20$.

Find ∇F and ∇g

$F = 18x^2 + 4yz - 16z + 597$ $\qquad g = 9x^2 + y^2 + 4z^2 - 20$

$\nabla F = 36x\,\mathbf{i} + 4z\,\mathbf{j} + (4y - 16)\mathbf{k}$ $\qquad \nabla g = 18x\,\mathbf{i} + 2y\,\mathbf{j} + 8z\,\mathbf{k}$

apply the Lagrange method in order to find the values of r, h, and λ.

$$\nabla F = \lambda \nabla g$$

$$36x\,\mathbf{i} + 4z\,\mathbf{j} + (4y - 16)\mathbf{k} = \lambda(18x\,\mathbf{i} + 2y\,\mathbf{j} + 8z\,\mathbf{k})$$

find λ.

$36x\,\mathbf{i} = \lambda\, 18x\,\mathbf{i}$

$\lambda = 2$ or $x = 0$

Find z in terms of y using $\lambda = 2$.

$4z\,\mathbf{j} = \lambda\, 2y\,\mathbf{j}$

$z = y$

Since $z = y$, Find y.

$(4y - 16)\mathbf{k} = \lambda\, 8z\,\mathbf{k}$ $\qquad y - z = 0$

$y - 4 = 2\lambda z$ $\qquad y - 4z = 4$

$y - 4z = 4$ $\qquad -3z = 4$

$z = -\frac{4}{3} = y$

$z = y$

$y - z = 0$

we need to Maximize $T = 18x^2 + 4yz - 16z + 596$

Subject to $9x^2 + y^2 + 4z^2 = 20 = g(x,y,z)$

By Lagrange Method, $\nabla T = \lambda \nabla g$

$36x = 18\lambda x$ (or) $2x = \lambda x \rightarrow ①$

$4z = 2\lambda y$ (or) $2z = \lambda y \rightarrow ②$

$4y - 16 = 8\lambda z$ (or) $y - 4 = 2\lambda z \rightarrow ③$

and $9x^2 + y^2 + 4z^2 = 20 \longrightarrow ④$

$① \Rightarrow (\lambda - 2)x = 0 \Rightarrow x = 0$ (or) $\lambda = 2$

Case-1: $\lambda = 2 \Rightarrow ② \Rightarrow z = y$

$③ \Rightarrow y - 4 = 4z$

$y - z = 0$

$y - 4z = 4$

$3z = 4$

$z = \frac{4}{3} = y$

$9x^2 = 20 - \frac{16}{9} - \frac{64}{9}$

$x^2 = \frac{180 - 80}{81}$

$x = \pm\frac{10}{9}$

we get two points, $\left(\frac{10}{9}, \frac{4}{3}, \frac{4}{3}\right), \left(-\frac{10}{9}, \frac{4}{3}, \frac{4}{3}\right)$

Temperature at both points

$T = 18 \times \frac{100}{81} + 4 \times \frac{4}{3} \times \frac{4}{3} - 16 \times \frac{4}{3} + 596$

$= \frac{200}{9} + \frac{64}{9} - \frac{256}{9} + 596$

$= \frac{264 - 256}{9} + 596 = 596 + \frac{8}{9}$.

Case(ii): $x = 0$: From ② & ③ $\frac{2z}{y} = \frac{y - 4}{2z}$

$4z^2 = y^2 - 4y$

From ④ $y^2 + 4z^2 = 20 \Rightarrow y^2 + y^2 - 4y = 20$

$y^2 - 2y - 10 = 0$

$y = 4.317, -2.317$

if $y = 4.317$

$4z^2 = y^2 - 4y \Rightarrow z = \pm 0.585$

if $y = -2.317 \Rightarrow z = \pm 1.913$

we get $\begin{bmatrix} 0 \\ 4.317 \\ \pm 0.5317 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2.317 \\ \pm 1.913 \end{bmatrix}$

at $\begin{bmatrix} 0 \\ 4.317 \\ 0.5317 \end{bmatrix} \Rightarrow T = 596.67$

at $\begin{bmatrix} 0 \\ 4.317 \\ -0.5317 \end{bmatrix} \Rightarrow T = 595.32$

at $\begin{bmatrix} 0 \\ -2.317 \\ 1.913 \end{bmatrix} \Rightarrow T = 547.66$

at $\begin{bmatrix} 0 \\ -2.317 \\ -1.913 \end{bmatrix} \Rightarrow T = 644.34$

The hottest point is $\begin{bmatrix} 0 \\ -2.317 \\ -1.913 \end{bmatrix}$

and coolest point is $\begin{bmatrix} 0 \\ -2.317 \\ 1.973 \end{bmatrix}$